Math 245C Lecture 29 Notes

Daniel Raban

June 7, 2019

1 Convolution of Distributions and Approximation of $W^{1,p}_{\text{loc}}$ Functions by C^∞ Functions

1.1 Convolution of distributions

If you solve $|Du| = 1$ with some boundary condition, it is unlikely that wou will find a solution in $C^1(\Omega)$. You will probably find a solution in $W^{1,1}_{\text{loc}}(\Omega)$. But we can approximate functions in $C^1(\Omega)$ by functions in $W^{1,1}_{\text{loc}}(\Omega)$. We can also approximate by functions in $C^\infty(\Omega)$. Oftentimes, we want to show that we have a solution in some bigger space and see if we can show it has extra properties that force it to be in a smaller, nicer space.

Let $\Omega \subseteq \mathbb{R}^d$ be an open set. If $\phi \in C_c^\infty(\Omega)$, we define $O_\phi = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\phi) \subseteq \Omega\}$. If $\psi \in L^1(O_\phi)$ is bounded, then

$$T(\psi * \phi) = \int_{O_\phi} \psi(y) T(\phi_y)\, dy.$$

for $T \in \mathcal{D}'(\Omega$ and $\phi_y(x) = \phi(x - y)$.

Given $j : A \subseteq \mathbb{R}^d \to \mathbb{R}$, we define $\tilde{h}L - A \to \mathbb{R}$ as $\tilde{j} = j(-x)$. If $T \in \mathcal{D}'(\mathbb{R}^d)$ and $j \in C_c^\infty(\mathbb{R}^d)$, we define $j * T : C_c^\infty(\mathbb{R}^d) \to \mathbb{R}$ as

$$j * T(\phi) = T(\tilde{j} * \phi).$$

Theorem 1.1. *Let $T \in \mathcal{D}'(\mathbb{R}^d)$, and let $j \in C_c^\infty(\mathbb{R}^d)$.*

1. *There exists $\psi \in C^\infty(\mathbb{R}^d)$ such that*

$$j * T(\phi) = \int_{\mathbb{R}^d} \phi(y)\psi(y)\, dy, \qquad \forall \phi \in C_c^\infty(\mathbb{R}^d)$$

   and so $j * T \in \mathcal{D}'(\mathbb{R}^d)$.

2. *Further assume $\int_{\mathbb{R}^d} j(x)\, dx = 1$, and set $j_\varepsilon = \varepsilon^{-d} j(x/\varepsilon)$ for $x \in \mathbb{R}^d$. Then $(j_\varepsilon * T)_\varepsilon$ converges to T in $\mathcal{D}'(\mathbb{R}^d)$ as $\varepsilon \downarrow 0$.*

Remark 1.1. This shows that we have an embedding from $C^\infty(\Omega)$ into $\mathcal{D}'(\Omega)$ and that this class of functions is dense in $\mathcal{D}'(\Omega)$.

Proof. Note that $O_{\tilde{j}} = \{y \in \mathbb{R}^d : y + \operatorname{supp}(\tilde{j}) \in \mathbb{R}^d\} = \mathbb{R}^d$. By the formula for distributions applied to convolutions, we get

$$j * T(\phi) = T(\tilde{j} * \phi) = \int_{O_{\tilde{j}}} \psi(y) T(\tilde{j}_y)\, dy.$$

Since $\tilde{j} \in C_c^\infty(\mathbb{R}^d)$, $y \mapsto T(\tilde{j}y)$ is of class C^∞.

For the second statement, if $\phi \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} j_\varepsilon * T(\phi) = \lim_{\varepsilon \to 0} T(\tilde{j}_\varepsilon * \phi) = T(\phi)$$

since $\tilde{j}_\varepsilon * \phi$ converges to ϕ in $C_c^\infty(\mathbb{R}^d)$. □

1.2 Approximation of $W^{1,p}_{\text{loc}}$ functions by C^∞ functions

Theorem 1.2. *Let $1 \leq p < \infty$, and let $f \in W^{1,p}_{\text{loc}}(\Omega)$. Then for every open, bounded $O \subseteq \mathbb{R}^d$ such that $\overline{O} \subseteq \Omega$, there exists $(f^k)_k \subseteq C^\infty(O)$ such that*

$$\lim_{k \to \infty} \|f - f^k\|_{W^{1,p}(O)} = 0.$$

Remark 1.2. This is equivalent to saying that $f_k \in C_0^\infty(O) \cap W^{1,p}_{\text{loc}}(O)$.

Proof. Let $\delta = \operatorname{dist}(\overline{O}, \Omega^c) > 0$. Let $j = C_c^\infty(\mathbb{R}^d)$ be such that $\int_{\mathbb{R}^d} j(x)\, dx = 1$ and $\operatorname{supp}(j) = \overline{B_1(0)}$. Set $j_\varepsilon(x) = \varepsilon^{-d} j(x/\varepsilon)$ for $0 < \varepsilon < \delta/3$. Note that $j_\varepsilon * f$, $j_\varepsilon * \nabla f$ are well-defined on O for these ε. We have $j_\varepsilon * f \in C^\infty(O)$ and that

$$0 = \lim_{\varepsilon \to 0} \|j_\varepsilon * f - f\|_{L^p(O)} = \lim_{\varepsilon \to 0} \|j_\varepsilon * \nabla f - \nabla f\|_{L^p(O)}.$$

Set $f^k = j_{1/k} * f$ to conclude the proof. □